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Numerical Evaluation for Computational Cost of CG-FMM on Typical Wiregrid Models

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Abstract The conjugate gradient-fast multipole method (CG-FMM) is one of the powerful methods for analysis of large-scale electromagnetic problems. It is also known that CPU time and computer memory can be reduced by CG-FMM but such computational cost of CG-FMM depends on shape and electrical properties of an analysis model. In this paper, relation between the number of multipoles and the number of segments in each group is derived from dimension of segment arrangement in four typical wiregrid models. Based on the relation and numerical results for these typical models, the CPU time per iteration and the computer memory are quantitatively discussed. In addition, the number of iteration steps, which is related to condition number of impedance matrix and analysis model, is also considered by a physical point of view.

Key words Method of moments (MoM), Fast multipole method (FMM), Conjugate gradient (CG).

1. Introduction

The Method of Moments (MoM) is one of the powerful techniques for numerical analysis of antennas and scatterers [1], [2]. Recently, iterative method such as the Conjugate Gradient (CG) method has been used to solve the MoM matrix equation in previous studies [3]- [8]. Unlike direct methods, the CPU time to solve the matrix equation by CG method is determined by number of iteration steps and CPU time per iteration. Therefore, computational cost of CG method is classified into three factors, i.e., the number of iteration steps, CPU time per iteration, and the computer memory. The number of iteration steps depends on the analysis model, while the CPU time per iteration and computer memory of CG method are both the order of $O(N^2)$.

In above three factors, the number of iterativon steps of CG method has been discussed in a few papers so far and it has been reported that the number of iteration steps strongly depends on the condition number of the impedance matrix $\mathbf{Z}$ [6]- [8]. In these papers, however, little attention is paid to relation between the number of iteration steps and analysis model, as well as relation between condition number of $\mathbf{Z}$ and analysis model.

On the other hand, it is known that other two factors except for the number of iteration steps, namely, the CPU time per iteration and computer memory of CG method can be reduced by the Fast Multipole Method (FMM) [9], [10]. It has been considered that the CPU time per iteration and computer memory of CG method combined with FMM (CG-FMM) are both $O(N^{1.5})$. However, the CPU time per iteration and computer memory in CG-FMM depend on many parameters for FMM as well as analysis model. Choice of parameters such as the number of multipoles or a group size for the two and three dimensional FMM have been discussed in a few papers [11], [12]. However, parameter choice based on these papers are not always correct in general since shape of analysis model, which is highly related to the number of multipoles L, has not been considered in these guidelines.

In this paper, four typical wiregrid models, which are electrically continuous or separated, and one-dimensional or two-dimensional, are introduced. It is shown that relation between the number of multipoles L and the number of segments in each group K is determined by dimension of segment arrangement in an analysis model, and the relation is used to discuss not only the CPU time per iteration but also the computer memory required for analysis. Based on the numerical simulation for these typical models, mutual relation among the number of iteration steps, condition number of $\mathbf{Z}$ and analysis model can be predicted universally from a physical point of view.

2. Principle of CG-FMM

2. 1 Cojugate Gradient Method

Matrix equation formulated by MoM is expressed by

$$\mathbf{ZI} = \mathbf{V}, \tag{1}$$

where $\mathbf{Z}$ is $N \times N$ impedance matrix, $\mathbf{V}$ is $N-$ dimensional known voltage vector, and $\mathbf{I}$ is $N-$ dimensional unknown current vector. The algorithm of CG method for solving Eq.(1) is summarized as follows [3]- [8].

CG method for MoM.

● After initial value of $\mathbf{I}$ is set to be $\mathbf{I}_0$, initial value of residual vector $\mathbf{R}_0$ and correction vector for solution $\mathbf{P}_0$ are calculated as

$$\mathbf{R}_0 = \mathbf{V} - \mathbf{Z}\mathbf{I}_0,$$
$$\mathbf{P}_0 = \mathbf{Z}^\dagger \mathbf{R}_0,$$

where $\mathbf{Z}^\dagger$ denotes the conjugate transpose of $\mathbf{Z}$.

● The same procedure is repeated as

$$\alpha_i = \frac{\langle \mathbf{Z}\mathbf{P}_{i-1}, \mathbf{R}_{i-1} \rangle}{\|\mathbf{Z}\mathbf{P}_{i-1}\|^2} = \frac{\left\|\mathbf{Z}^\dagger \mathbf{R}_{i-1}\right\|^2}{\|\mathbf{Z}\mathbf{P}_{i-1}\|^2},$$
$$\mathbf{I}_i = \mathbf{I}_{i-1} + \alpha_i \mathbf{P}_{i-1},$$
$$\mathbf{R}_i = \mathbf{V} - \mathbf{Z}\mathbf{I}_i = \mathbf{R}_{i-1} - \alpha_i \mathbf{Z}\mathbf{P}_{i-1}.$$

● When $\|\mathbf{R}_i\| < \epsilon \|\mathbf{V}\|$, the iteration is stopped, β_i and $\mathbf{P}_i$ is calculated as

$$\beta_i = \frac{\left\|\mathbf{Z}^\dagger \mathbf{R}_i\right\|^2}{\|\mathbf{Z}^\dagger \mathbf{R}_{i-1}\|^2},$$
$$\mathbf{P}_i = \mathbf{Z}^\dagger \mathbf{R}_i + \beta_i \mathbf{P}_{i-1},$$

where α_i and β_i are correction coefficients for $\mathbf{I}_{i-1}$ and $\mathbf{P}_{i-1}$, respectively, and ϵ is an error control parameter for the solution.

In the above algorithm, matrix-vector multiplications are carried out twice in each iteration and CPU time for the matrix-vector multipication is $O(N^2)$. In addition, computer memory for storing $\mathbf{Z}$ is also $O(N^2)$.

2. 2 Fast Multipole Method

FMM is a method based on the expression of the scalar Green's function using the addition theorem [9], [10]. FMM can reduce both the CPU time for the matrix-vector multiplication and the computer memory of CG method to $O(N^{1.5})$ when $M = K = \sqrt{N}$ where M is the number of groups and K is the number of segments in each group. The matrix-vector multiplication in each step of CG-FMM can be collectively carried out by grouping scheme based on the addition theorem. In addition, mutual impedance between far segments are calculated by the addition theorem at every time of the matrix-vector multiplication in CG-FMM and the impedance matrix $\mathbf{Z}$ is not required to be stored.

Using FMM, mutual impedance between far segments are represented by

$$Z^{\text{far}}_{mkm'k'} \approx \frac{\omega\mu_0 k_0}{(4\pi)^2} \int_0^{2\pi}\int_0^{\pi} \mathbf{s}_{mk}(\hat{\mathbf{k}}) T_{mm'} \mathbf{s}^*_{m'k'}(\hat{\mathbf{k}}) \sin\theta d\theta d\phi, \tag{2}$$

where $\mathbf{s}_{mk}(\hat{\mathbf{k}})$, $\mathbf{s}_{m'k'}(\hat{\mathbf{k}})$, and $T_{mm'}$ are called "radiation function", "receiving function", and "transfer function", respectively. Detailed explanation of these functions is rather lengthy and omitted here (see [14]). Subscript m and $m'(= 1, 2, ..., M)$ are the observation group number and the source group number, respectively. k and $k'(= 1, 2, ..., N/M(= K))$ represent segment number in observation and source group, respectively. Computational cost of CG-FMM depends on the number of groups M and the number of segments K in each group.

Another important parameter is the number of multipoles L, which is truncation number of infinite series $T_{mm'}$. The number of multipoles L is given by the following empirical formula

$$L = k_0 D_{\max} + \alpha_L \ln(k_0 D_{\max} + \pi), \tag{3}$$

where $D_{\max}$ is the maximum diameter of the groups and $\alpha_L (\sim 0 - 10)$ is the error control parameter for the transfer function. In addition, L is also used as the points of double numerical integration in Eq. (2). In order to obtain accurate results, L point Gauss-Legendre integration in θ and $2L$ point trapezoidal integration in ϕ are applied to Eq. (2). Namely, each component of $\mathbf{s}_{mk}(\hat{\mathbf{k}})$, $\mathbf{s}_{m'k'}(\hat{\mathbf{k}})$ as well as $T_{mm'}$ are stored as $L \times 2L = 2L^2$ discreate data in numerical calculation.

Based on Eq. (2), the matrix-vector multiplication for Z^{far} in CG-FMM is calcuated by three steps, namely, the aggregation step (Step 1), the translation step (Step 2), and the disaggregation step (Step 3). The order of CPU time for the multiplication of Z^{near}, which denotes the mutual impedance between near segments, and that of Z^{far} are shown in Table 1. The computer memory for CG-FMM is also tabulated in Table 2. In general, the computational cost of $O(N^{1.5})$ is realized when $M = K = \sqrt{N}$.

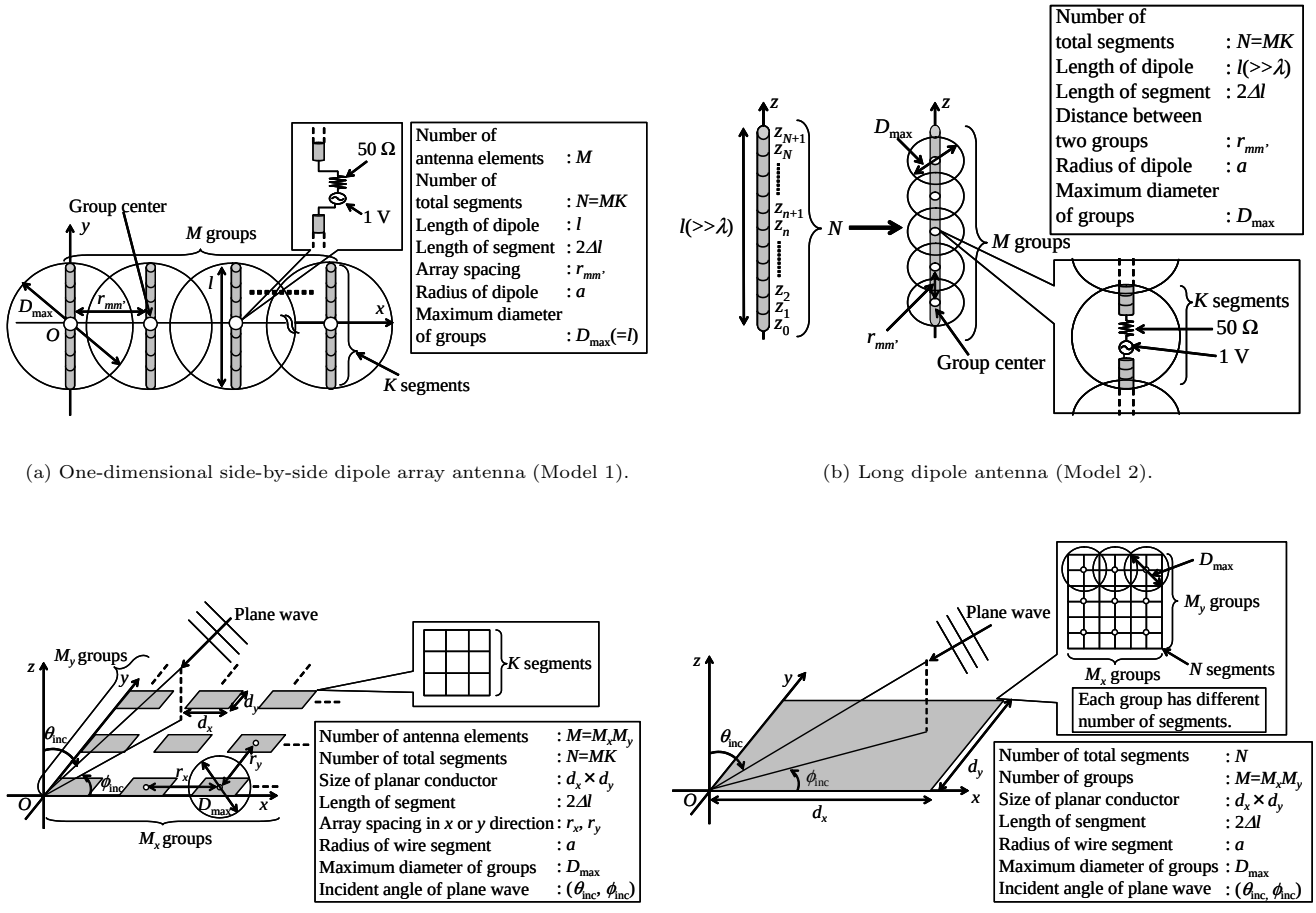
Table 1 CPU time for matrix-vector multiplication in CG-FMM.

Type of multiplication	Step	CPU time
Multiplication for Z^{near}		$O(MK^2)$
Multiplication for Z^{far}	Step 1	$O(2NL^2)$
	Step 2	$O(2M^2L^2)$*
	Step 3	$O(2NL^2)$

* CPU time for Step 2 is dominant in the most cases of large-scale models.

3. Typical Models

Four typical models described in the present paper are shown in Fig. 1. Fig. 1(a) shows one-dimensional side-by-side dipole array antenna (Model 1), where dipole element,



(a) One-dimensional side-by-side dipole array antenna (Model 1).

(b) Long dipole antenna (Model 2).

(c) Two-dimensional array of planar scatterers (Model 3).

(d) Planar scatterer (Model 4).

Figure 1 Four typical wiregrid models.

Table 2 Computer memory for CG-FMM.

Stored contents	Computer memory
$\mathbf{V}, \mathbf{I}, \mathbf{P}, \mathbf{R}$	$O(N)$
$W(\theta)$	$O(L)$
$Z^{\text{near}}_{mkm'k'}$	$O(MK^2)$
$\mathbf{s}_{mk}(\hat{\mathbf{k}})$ and $\mathbf{s}_{m'k'}(\hat{\mathbf{k}})$	$O(2NL^2)$
$T_{mm'}(k_0 r_{mm'}, \hat{\mathbf{k}} \cdot \hat{\mathbf{r}}_{mm'})$	$O(2M^2L^2)$

having length l and radius a is arrayed periodically. Each element is divided into K wire segments.

Fig. 1(b) shows long dipole antenna (Model 2), having length l ($\gg \lambda$) and radius a. The dipole antenna is divided into N wire segments.

Fig. 1(c) shows two-dimensional array of conducting planar scatterers (Model 3). The planar element with a size of $d_x \times d_y$ is arrayed periodically. Each element is divided into K wire segments with radius a.

Fig. 1(d) shows a planar conductor (Model 4). A planar element with a size of $d_x \times d_y$ is divided into N wire segments with radius a.

In models $1-3$, periodicity of the group and segment location are both satisfied. In the model 4, the group location is only periodic in this model. Since the uniform grouping for segments is difficult, non-uniform grouping scheme and versatile program using CG-FMM is used for analysis of the model 4.

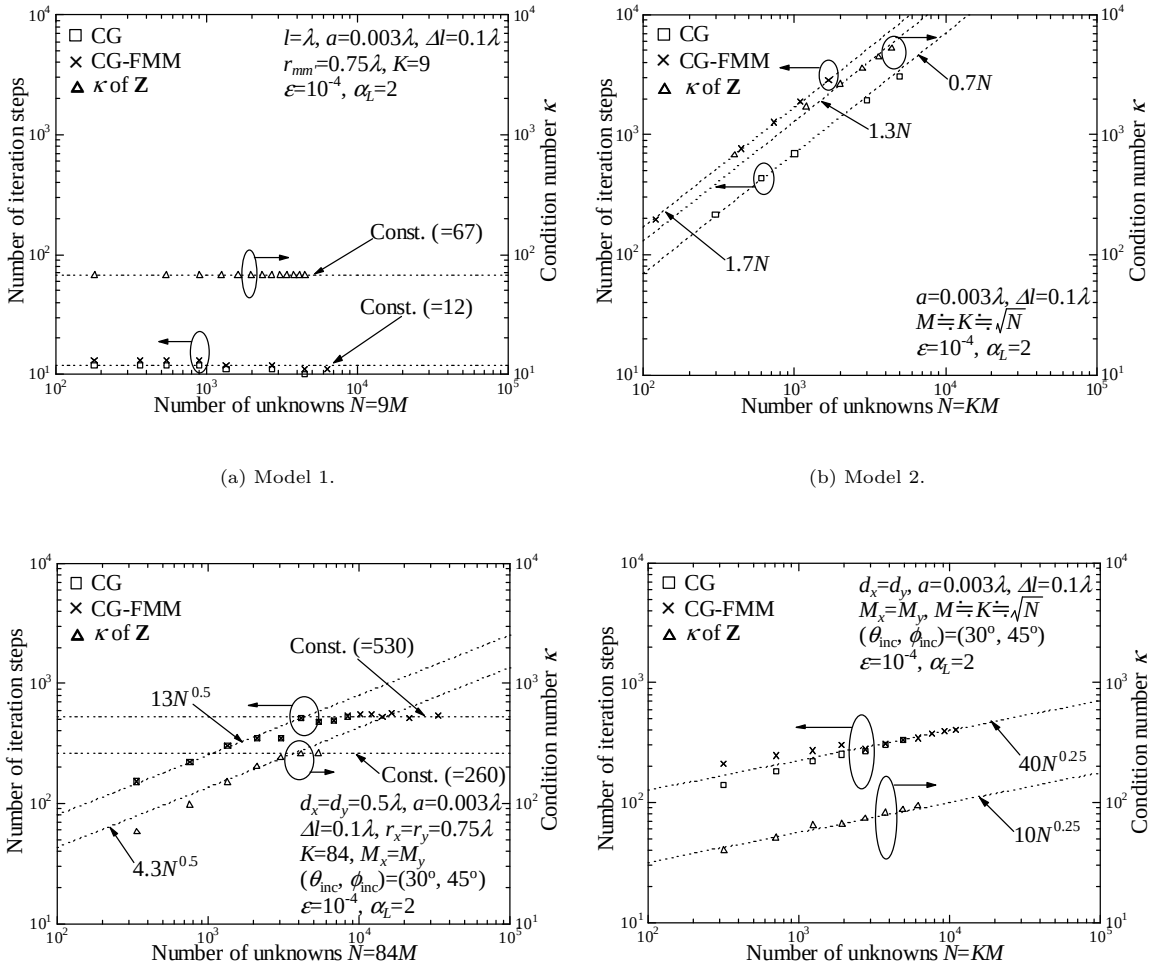
Table 3 Relation between L and K.

Segment arrangement	Relation between L and K
1D (Models 1 and 2)	$L \propto K$
2D (Models 3 and 4)	$L^2 \propto K$

Table 4 Computer memory reduction using periodicity in CG-FMM.

Type of periodicity in analysis model	Required computer memory		
	$Z^{\text{near}}_{mkm'k'}$	$T_{mm'}$	$\mathbf{s}_{mk}$, $\mathbf{s}_{m'k'}$
None	$O(MK^2)$	$O(2M^2L^2)$	$O(2NL^2)$
Group location	$O(MK^2)$	$O(2ML^2)$	$O(2NL^2)$
Group and segment location	$O(K^2)$	$O(2ML^2)$	$O(2KL^2)$

Relation between the number of multipoles L and segments in each group K is shown in Table 3. In general, the number of segments K is proportional to $(D_{\max}/\lambda)^2$ for two-dimensional arrangement, while K is proportional



(a) Model 1.

(b) Model 2.

(c) Model 3.

(d) Model 4.

Figure 2 Number of iteration steps and condition number of **Z**.

to $(D_{\max}/\lambda)$ for one-dimensional arrangement. Substituting the relation between K and $D_{\max}$ into Eq. (3), the relation between L and K shown in Table 3 can be easily derived and shown in Table 3.

The use of periodicity in CG-FMM can reduce the computer memory required for analysis as shown in Table 4. By using Tables 2 - 4, the computer memory required for CG-FMM corresponding to each analysis model can be easily estimated. In a similar way, CPU time per iteration in CG-FMM can be also predicted by Tables 1 and 3.

4. Numerical Results

Numerical analysis of four typical models is carried out using the Richmond's MoM [2]. Dell Precision PWS 380 with 2 GB RAM is used for all numerical calculation. $\epsilon = 10^{-4}$ is used for the convergence criterion of CG-FMM and $\alpha_L = 2$ is employed to Eq. (3) for sufficient truncation number. In addition, the self impedance and the mutual impedance between segments in adjacent groups are calculated by the MoM rather than by FMM.

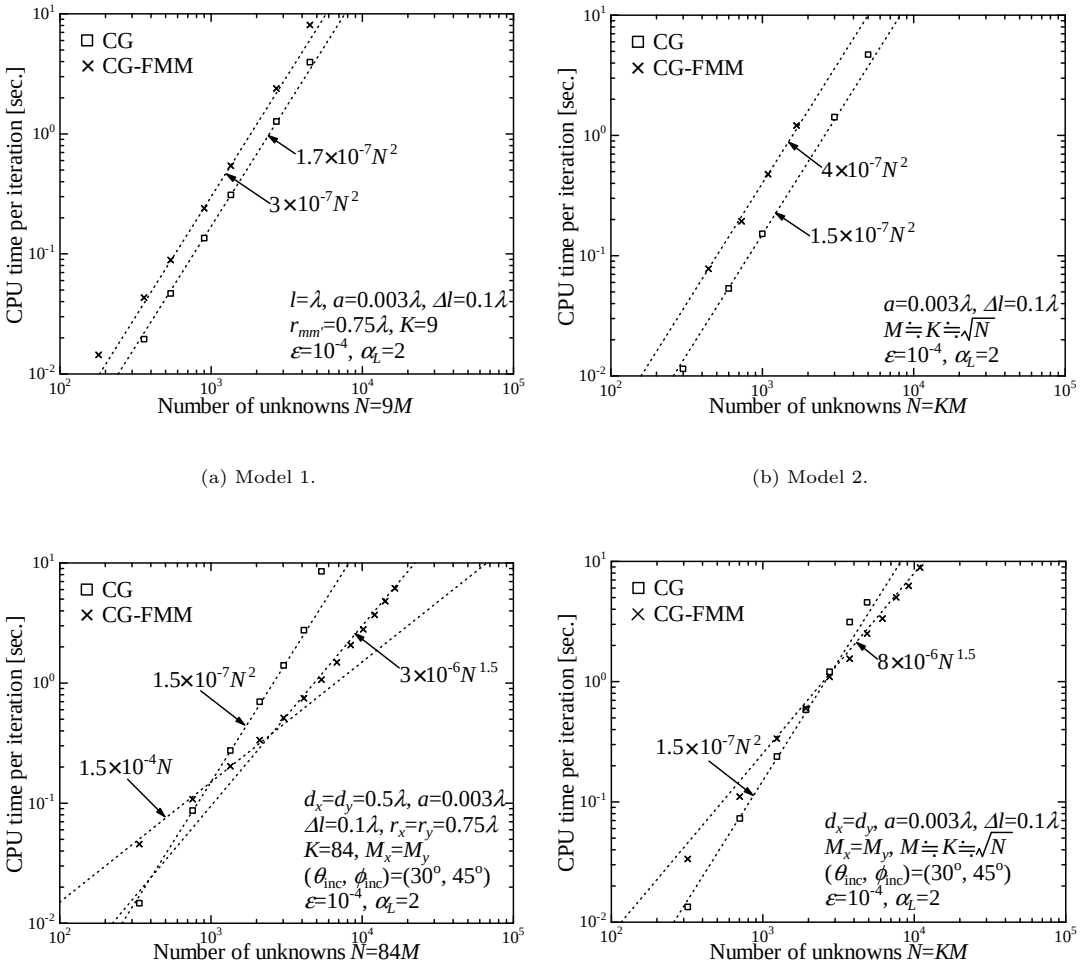
4.1 Number of Iteration steps and Condition Number

The number of iteration steps required for CG method and CG-FMM and the condition number κ of **Z** for four models are shown in Fig. 2. The condition number κ of **Z** is defined by

$$\kappa = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}. \tag{4}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalue of $\mathbf{Z}^{\dagger}\mathbf{Z}$, respectively. As shown in Fig. 2, the tendency of the condition number κ is almost the same to that of the number of iteretion steps determined by error control parameter. From those results, it is found that the condition number is highly related to that of the number of iteretion steps.

Large κ means that the impedance matrix **Z** is ill-conditioned and convergence of the solution obtained in iterative procedure is slow. On the other hand, small κ means that the impedance matrix **Z** is well-conditioned and convergence of the solution is fast.



(a) Model 1.

(b) Model 2.

(c) Model 3.

(d) Model 4.

Figure 3 CPU time per iteration for analysis.

From a physical point of view, condition number κ indicates how the solution $\mathbf{I}$ in Eq. (1) is sensitive to the values of the elements in the impedance matrix $\mathbf{Z}$. When electrical connection between segments in the model is strong due to linear-connection of segments or strong mutual coupling such as Models 2 and 3, the condition number κ increases as shown in Figs. 2(b) and 2(c), and $\mathbf{I}$ is sensitive to $\mathbf{Z}$. On the other hand, when electrical connection between segments in the model is weak due to small mutual coupling or grid-connection of segments such as Models 1 and 4, condition number κ is small as shown in Figs. 2(a) and 2(d), and $\mathbf{I}$ is not sensitive to $\mathbf{Z}$. Constant κ shown in Figs. 2(a) and 2(c) means that mutual coupling effect to an element does not change anymore even when M increases.

4. 2 CPU Time per Iteration

The CPU time per iteration for analysis of four typical models is shown in Fig. 3. It is found that the CPU time per iteration of CG-FMM is $O(N^2)$ for the Models 1 and 2, while that for the Models 3 and 4 is reduced to $O(N^{1.5})$. Using $L \propto K$ as shown in Table 3, it can be derived from Table 1 that the CPU time per iteration for analysis of one-dimensional models is $O(N^2)$, which is independent of the value of M, because the major CPU time of Step 2 in Table 1 is $O(N^2)$. By these observations, it is concluded that the CPU time per iteration to analyze an antenna which has one-dimensional segment arrangement can not be reduced by CG-FMM.

4. 3 Computer Memory

The order of the computer memory required for analysis of each model can be theoretically derived by Tables 2 - 4 without running programs and is tabulated in Table 5. From the Table 5, it is found that the computer memory required for the analysis of any models is smaller than $O(N^2)$ in CG-FMM. For the array antenna/scatterer such as the Models 1 and 3, the computer memory for $\mathbf{s}_{mk}$ and $\mathbf{s}_{m'k'}$ is dominant for the total computer memory when M is very small. On the other hand, the computer memory for $T_{mm'}$ is dominant for the total computer memory when M is larger than K. Therefore, the order of the computer memory required for the analysis varies as $O(K^3) \rightarrow O(N)$ for Model 1 and

Table 5 Computational cost of CG-FMM for four typical wiregrid models.

Item		Model 1	Model 2	Model 3	Model 4
Properties of analysis model	Segment arrangement	1D	1D	2D	2D
	Segment connection	Linearly connected	Linearly connected	Grid connected	Grid connected
	Number of elements	Array	Single	Array	Single
Type of periodicity used in CG-FMM	Group location	Used	Used	Used	Used
	Segment location	Used	Used	Used	Not used
Relation between L and K		$L \propto K$	$L \propto K$	$L^2 \propto K$	$L^2 \propto K$
Parameter setting on grouping		$K = \text{Const.}$	$M = K = \sqrt{N}$	$K = \text{Const.}$	$M = K = \sqrt{N}$
Computational cost	Number of iteration steps	Const.	$O(N)$	$O(N^{0.5}) \rightarrow \text{Const.}$	$O(N^{0.25})$
	CPU time per iteration	$O(N^2)$	$O(N^2)$	$O(N) \rightarrow O(N^{1.5})$	$O(N^{1.5})$
	Total CPU time	$O(N^2)$	$O(N^3)$	$O(N^{1.5})$	$O(N^{1.75})$
	Computer memory	$O(K^3) \rightarrow O(N)$	$O(N^{1.5})$	$O(K^2) \rightarrow O(N)$	$O(N^{1.5})$

$O(K^2) \rightarrow O(N)$ for Model 3.

The order of the number of iteration steps, CPU time per iteration, total CPU time and computer memory is summarized in Table 5.

5. Conclusion

In this paper, relation between the computational cost of CG-FMM and the analysis model was quantitatively evaluated by numerical simulation for four typical wire antennas or wiregrid models. It is found that the number of iteration steps required for the analysis by CG-FMM depends on the condition number of $\mathbf{Z}$. From discussion of relation between the condition number κ and electrical properties of the models, it is shown that the condition number κ becomes large due to linear-connection of segments or strong mutual coupling between elements. Similarly, it is shown that the condition number κ becomes small due to grid-connection of segments exist or weak mutual coupling between elements. The computer memory required for the analysis are reduced from $O(N^2)$ for all models by the use of CG-FMM. On the other hand, it is shown that the CPU time per iteration can be reduced only for models which has two-dimensional segment arrangement in each group.

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