LETTER

Convergence of SOR in MoM Analysis of Array Antenna

Qiang CHEN[†a)], ***Member***, **Qiaowei YUAN**[††], ***Nonmember***, ***and*** **Kunio SAWAYA**[†], ***Member***

SUMMARY Convergence of the iterative method based on the successive overrelaxation (SOR) method is investigated to solve the matrix equation in the moment analysis of array antennas. It is found this method can be applied to the sub domain method of moments with fast convergence if the grouping technique is applied and the over-relaxation parameter is properly selected, and the computation time for solving the matrix equation can be reduced to be almost proportional to the second power of the number of unknowns.
key words: *antenna, array antennas, method of moments, matrix equation, SOR, iterative method*

1. Introduction

It is required to develop an efficient analysis method to investigate the characteristics of periodic structures such as array antennas and the metamaterials consisting of a large amount of small resonant particles which have some interesting electromagnetic properties.

The method of moments (MoM) is one of the efficient methods for the electromagnetic analysis of array antennas. However, when the direct method such as the Gauss-Jordan method is employed to solve the matrix equation appearing in the MoM, the CPU time is proportional to the third power of the number of unknowns. Therefore, the computational cost to solve the matrix equation has to be reduced to analyze large-scale array antennas with a large number of unknowns.

Iterative methods such as the Conjugate Gradient (CG) method are efficient to solve the linear matrix equations [1]. Several methods to accelerate the matrix-vector multiplication for the iterative methods have been proposed, such as the fast multipole method (FMM) [2], the multilevel fast multipole algorithm (MLFMA) [3] and the fast inhomogeneous plane wave algorithm [4].

The iterative algorithm based on the Gauss-Seidel scheme has been proposed for the MoM analysis of array antennas by the present authors and it has been shown that the CPU time to solve the matrix equation is almost proportional to the second power of the number of unknowns [5].

The SOR method is an iterative method which applies extrapolation to the Gauss-Seidel method [6], [7]. This extrapolation modifies the iterative step by introducing the so called over-relaxation parameter to accelerate the convergence rate. Therefore, the SOR method usually costs less CPU time than the Gauss-Seidel method if the over-relaxation parameter is selected properly.

In this research, convergence of the iterative algorithm based on the SOR technique is investigated to solve the matrix equation in the MoM analysis of array antennas to clarify the effect of the over-relaxation parameter on the convergence. Computational cost of the method is shown by some numerical examples.

2. Iteration Scheme

The linear array antenna shown in Fig. 1 is used as the analysis model, where the array contains N dipole elements having length of $2h$ and radius of a, and array spacing is d. Each dipole element is divided into M overlapped dipole segments acting as the piece-wise sinusoidal basis and test functions for the sub domain MoM analysis [8]. Once the impedance matrix $[Z]$ with dimension of $N_T \times N_T$, which includes the self and mutual impedance between the divided segments, is built up, the unknown current vector $[I]$ can be obtained by solving the matrix equation $[Z][I] = [V]$, where $N_T = M \times N$ and $[V]$ is the incident voltage vector with dimension of N_T.

In order to apply the SOR method to solving the matrix equation, the matrix $[Z]$ is split into $[S]$ and $[T]$ so that the matrix equation becomes

$$[S][I] = -[T][I] + [V], \tag{1}$$

where $[S]$ is the lower-left triangular part including the diagonal elements of $[Z]$, and $[T]$ is the upper-right triangular part excluding the diagonal elements.

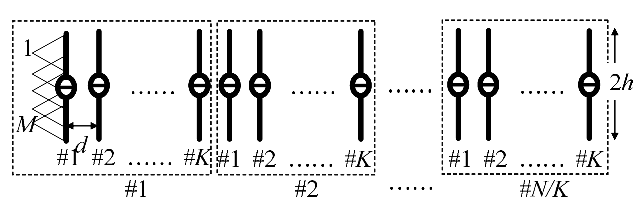

Fig. 1 A linear dipole array antenna which is divided into N/K groups and each group consists of $K(\geq 1)$ neighboring array elements. Each dipole element is divided into M overlapped dipole segments acting as the piece-wise sinusoidal basis and test functions for MoM analysis.

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[†]The authors are with the Department of Electrical Communications, Faculty of Engineering, Tohoku University, Sendai-shi, 980-8579 Japan.
[††]The author is with Intelligent Cosmos Research Institute (ICR), Sendai-shi, 989-3204 Japan.
a) E-mail: chenq@ecei.tohoku.ac.jp
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It has been pointed out that the iteration to solve the above matrix equation appearing in the MoM analysis by using the conventional Gauss-Seidel iterative method usually has poor convergence. In order to improve the convergence characteristics of the iteration, a grouping technique has been proposed in the previous study [5]. The array antenna to be solved is divided into groups and each group consists of one or several neighboring array elements, so that the impedance matrix can be decomposed into a number of sub matrices corresponding to the groups of the array elements. For example, the analysis model is divided into groups and each group contains $K(\geq 1)$ array elements as shown in Fig. 1, where the total array elements are divided into N/K groups completely. Therefore, the sub matrices are the basic iteration units rather than the matrix element in the original SOR iteration method, and the iterating procedure is expressed by:

$$\begin{aligned}\left[\bar{\zeta}\right]_i^{(l+1)} &= \left[\bar{I}\right]_i^{(0)} - \left[\overline{Z}\right]_{ii}^{-1}\left[\sum_{j=1}^{i-1}\left[\overline{Z}\right]_{ij}\left[\bar{I}\right]_j^{(l+1)}\right.\\ &\quad\left.+\sum_{j=i+1}^{N/K}\left[\overline{Z}\right]_{ij}\left[\bar{I}\right]_j^{(l)}\right]^H,\\ \left[\bar{I}\right]_i^{(l+1)} &= \left[\bar{I}\right]_i^{(l)} + \omega\left(\left[\bar{\zeta}\right]_i^{(l+1)} - \left[\bar{I}\right]_i^{(l)}\right),\\ i &= 1, 2, \cdots, N/K;\, l = 0, 1, \cdots, L,\end{aligned} \tag{2}$$

where $\left[\bar{I}\right]_i$ is the current vector of the ith group with dimension of MK and $\left[\bar{\zeta}\right]_i$ is a temporary vector for containing $\left[\bar{I}\right]_i$. $\left[\overline{Z}\right]_{ij}$ is the sub matrix composed of the self and mutual impedance of the dipole segments between two groups i and j, and its inversed matrix $\left[\overline{Z}\right]_{ii}^{-1}$ is evaluated by using the Gauss-Jordan method. ω is the overrelaxation parameter, which should be properly determined to accelerate the convergence rate. The SOR method becomes the Gauss-Seidel method if ω is unity. The superscript l indicates the step number of the iteration. The iteration step is repeated until the criterion

$$|I_k^{(L)} - I_k^{(L-1)}|/|I_k^{(L-1)}| \leq \epsilon \tag{3}$$

is satisfied for all the dipole segments k $(1 \leq k \leq N_T)$, where I_k is the current on the kth dipole segment and ϵ is a small number, which is eual to 1×10^{-8} in the following calculation. The initial $[\bar{I}]_i^{(0)}$ contains the current on the ith group without considering the mutual coupling from the other groups, and is given by

$$[\bar{I}]_i^{(0)} = [\bar{Z}]_{ii}^{-1}[\bar{V}]_i, \quad i = 1, 2, \cdots, N/K. \tag{4}$$

where $[\bar{V}]_i$ is the incident voltage vector of the ith group. In the iteration scheme, the unknown current on each dipole segment, which starts from the initial value which does not include the coupling from neighboring groups, converges to a value including the mutual coupling step by step.

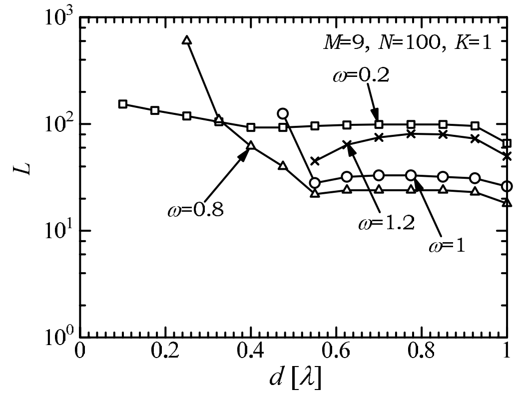

Fig. 2 Iteration steps required to perform Eq. (2) as function of d for various ω, when M=9, N=100 and K=1.

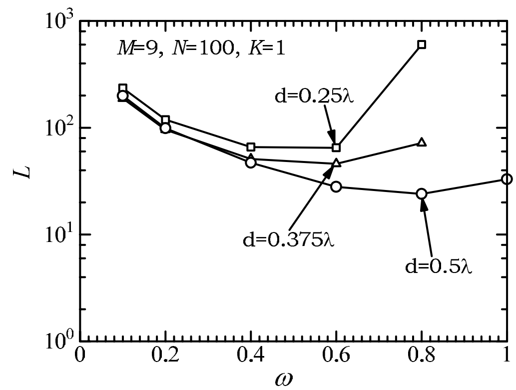

Fig. 3 Iteration steps required to perform Eq. (2) as function of ω for various array spacing d, when M=9, N=100 and K=1.

3. Convergence of Iteration

In this section, convergence of the iterative algorithm given by Eq. (2) is investigated numerically by solving the linear dipole array shown in Fig. 1. The parameters of the array elements are assumed to be $2h$=0.5 λ and $a = 2.5 \times 10^{-3}\,\lambda$, where, λ is the wavelength in free space.

Figure 2 shows the required iteration steps L as a function of the array spacing d for various ω, where M=9, N=100 and K=1. It is indicated that a relatively large ω, for example, ω=0.8 is better than ω=0.2 when d is larger than about 0.3 λ. However, when d becomes smaller than 0.5 λ, the value of L increases rapidly for ω=0.8, while it does not increase obviously for $\omega = 0.2$. It is also found that selecting ω larger than 1 causes divergence when d is smaller than about 0.5 λ. The convergence due to the value of ω is further investigated by evaluating the required iteration steps L when ω is changed at a smaller step and the results are shown in Fig. 3. When the array spacing is relatively large (d=0.5 λ), ω=0.8 is suitable. However, when the array spacing is reduced to d=0.375 and 0.25 λ, a smaller ω ranged from 0.4 to 0.6 is a better choice.

When K is increased up to 5, as shown in Fig. 4, the convergence property in the case of relatively large ω and

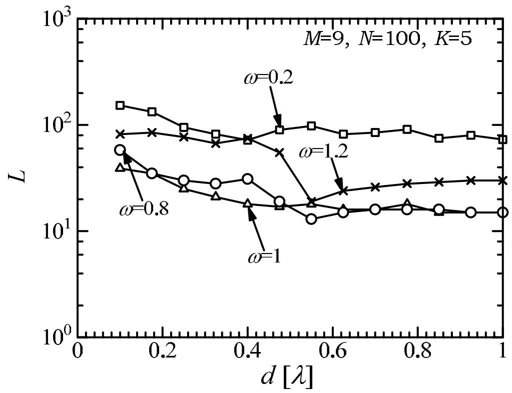

Fig. 4 Iteration steps required to perform Eq. (2) as function of d for various ω, when M=9, N=100 and K=5.

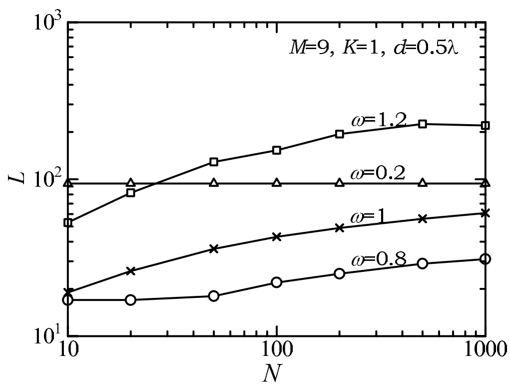

Fig. 5 Iteration steps required to perform Eq. (2) versus number of dipole array elements for various ω.

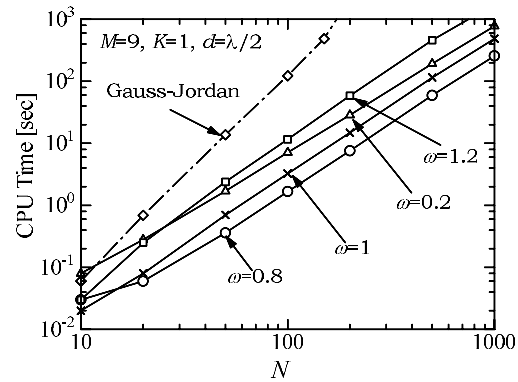

Fig. 6 CPU time as function of N for solving matrix equation of half wavelength dipole antenna array in the case of K = 1.

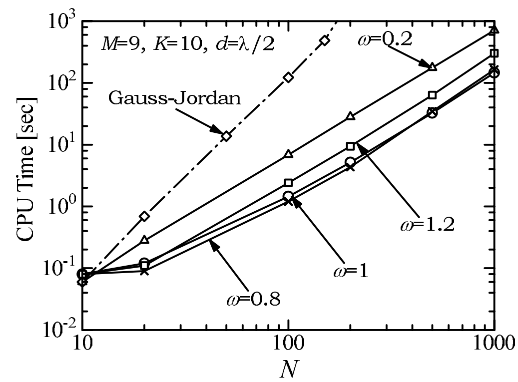

Fig. 7 CPU time as function of N for solving matrix equation of half wavelength dipole antenna array in the case of K = 10.

small d is much improved. Therefore, it can be said that it is effective to obtain a good convergence by using a relatively large ω and enlarging the size of groups, especially for the case of a small array spacing.

The required number of iteration steps versus the total number of the dipole array elements is shown in Fig. 5 in the case of M=9, K=1 and d=0.5 λ. The curve of ω=0.2 becomes almost flat, and the curves of other ω increase a little bit when N increases, which means the computational cost consumed by the present method is approximately proportional to N^2. However, ω=0.8 is the most suitable value to obtain a good convergence in this case.

It should be noted that the over-relaxation parameter in the original SOR method is usually larger than 1 to enlarge the iterative step in order to improve the convergence. However, the above numerical studies show that this parameter used in the present method should be usually less 1 to make the convergence being stable.

The CPU time for solving the matrix equation versus the number of array elements N is shown in Fig. 6 and Fig. 7. The curve of the Gauss-Jordan method is also plotted for comparison. When each group contains one dipole element, ω=0.8 is the most effective in saving the CPU time, while ω=1.2 is the most time-consuming as shown in Fig. 5. If K = 10 which means that each group contains 10 array elements, the CPU time is reduced for the relatively large ω between 0.8 and 1.2, while it does not change obviously for the case of ω=0.2. These two figures also show that the CPU time is almost proportional to N^2. Compared with N^3 required by the Gauss-Jordan method, the computational cost saving effect of the SOR method is significant.

4. Conclusions

Convergence of the iterative algorithm based on SOR method has been investigated to solve the matrix equation of the MoM analysis for array antennas. It has been shown that convergence of the SOR method can be improved if the grouping technique is applied and the over-relaxation parameter ω is selected properly. From the numerical results, it can be concluded that a relatively large ω, usually less than 1, can accelerate the iteration when the mutual coupling is small between the neighboring groups, but might make the iteration unstable and degrade the convergence if the mutual coupling is strong, or if the array has a large number of elements. The grouping technique can reduce the mutual coupling between the groups and make the convergence being stable. Therefore, a relatively large ω together with the grouping technique can be expected to accelerate the convergence rate of the iteration. It has been demonstrated that the CPU time is approximately proportional to the second power of the number of unknowns when the number is large enough, which has been greatly reduced compared with a direct method such as the Gauss-Jordan method. How to choose the ω properly and how to apply the grouping technique to gather the array elements into groups properly mainly depend on the geometry of the array elements, array spacing, input voltage at each array element and so on, and is somewhat difficult to be determined. These problems are targets of the future study.

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